We shall assume that the nuclei can be treated classically, so that

$$n_{++}=n_{+0}\exp(-\lambda Ze\psi_+/kT). \qquad (5)$$

The electrons must, however, be described by Fermi-Dirac statistics. In the Thomas-Fermi approximation, we have[5]

$$n_{-+}=\frac{8\pi}{h^3}\int_0^\infty\frac{p^2dp}{1+\exp[(p^2/2m-\lambda e\psi_+-\mu)/kT]}$$
$$=4\pi(2mkTh^{-2})^{\frac{3}{2}}I_{\frac{1}{2}}(\eta_+), \qquad (6)$$

where

$$I_m(\eta)=\int_0^\infty y^m(1+e^{y-\eta})^{-1}dy, \qquad (7)$$

$$\eta_+=(\lambda e\psi_++\mu)/kT, \qquad (8)$$

and the free-electron chemical potential μ is such that

$$n_{-0}=4\pi(2mkTh^{-2})^{\frac{3}{2}}I_{\frac{1}{2}}(\eta_\infty), \qquad (9)$$

$$\eta_\infty=\mu/kT. \qquad (10)$$

For purposes of numerical calculation, it is convenient to introduce the following units of length and energy[6]

$$r_\lambda=\frac{h^2}{4\pi^2m\lambda^2e^2}\left(\frac{9\pi^2}{128Z}\right)^{\frac{1}{3}}=\frac{0.468479\times10^{-8}\text{cm}}{\lambda^2Z^{\frac{1}{3}}}, \qquad (11)$$

and

$$\theta_\lambda=32m\lambda^4e^4h^{-2}=22.0532\lambda^4\text{ev}, \qquad (12)$$

and also the quantities

$$x=r/r_\lambda, \qquad \theta=kT/\theta_\lambda, \qquad (13)$$

$$4\epsilon=(6/\pi^2Z^2)^{\frac{1}{3}}=0.84713084Z^{-\frac{2}{3}}, \qquad (14)$$

$$\eta_+=\theta^{-1}(4\epsilon)^{-2}(\phi_+/x). \qquad (15)$$

Combining all the above, the Poisson equation (3) reduces to

$$\phi_+''(x)=\tfrac{3}{2}(4\epsilon)^3\theta^{\frac{3}{2}}x\{I_{\frac{1}{2}}(\eta_+)-I_{\frac{1}{2}}(\eta_\infty)\exp[-Z(\eta_+-\eta_\infty)]\}, \qquad (16)$$

with boundary conditions

$$\phi_+(0)=1,$$
$$\lim_{x\to\infty}\phi_+(x)=x(\phi/x)_\infty=x\phi_\infty'. \qquad (17)$$

For given temperature, bulk density of material, and value of λ, the procedure is as follows: n_{-0} can be readily calculated from the bulk density, θ found from (13), $I_{\frac{1}{2}}(\eta_\infty)$ from (9), and η_∞ from the tables and asymptotic expansions for $I_{\frac{1}{2}}$ given by McDougall and Stoner.[7] The differential equation (16) can then be integrated to give $\phi_+(x)$, and hence $\eta_+(x)$ from (15). The distribution of particles about a given nucleus then follows from (6) and the equivalent of (5)

$$n_{++}=n_{+0}\exp[-Z(\eta_+-\eta_\infty)]. \qquad (18)$$

The net charge surrounding the given nucleus is

$$q_+=4\pi r_\lambda^3\int_0^\infty(\lambda Zen_{++}-\lambda en_{-+})x^2dx. \qquad (19)$$

Using (6), (18), and the differential equation (16), this can be written

$$q_+=-\lambda Ze\int_0^\infty\phi_+''xdx$$
$$=-\lambda Ze[x\phi_+'-\phi_+]_0^\infty=-\lambda Ze, \qquad (20)$$

from the boundary conditions (17). Thus q_+ is, as it should be, the negative of the charge on the given nucleus.

b. Particle Distributions about an Electron

Singling out a specific electron, let the average electrostatic potential (due to all charges, including the electron in question) and the average charge density about this electron be, respectively, $\psi_-(r)$ and

$$\rho_-(r)=\lambda Zen_{+-}(r)-\lambda en_{--}(r). \qquad (21)$$

These quantities are related through the Poisson equation

$$\Delta\psi_-=-4\pi\rho_-=-4\pi\lambda e(Zn_{+-}-n_{--}), \qquad (22)$$

with boundary conditions

$$\lim_{r\to0}r\psi_-(r)=-\lambda e$$
$$\lim_{r\to\infty}\psi_-(r)=0. \qquad (23)$$

For a neutral plasma, it follows from symmetry considerations that the distribution of positive charge about an electron must be identical in form to the distribution of negative charge about a nucleus. Thus from (6),

$$n_{+-}=Z^{-1}n_{-+}=4\pi Z^{-1}(2mkTh^{-2})^{\frac{3}{2}}I_{\frac{1}{2}}(\eta_+). \qquad (24)$$

Letting

$$\eta_-=(\lambda e\psi_-+\mu)/kT=\lambda e\psi_-/kT+\eta_\infty, \qquad (25)$$

then analogously to (6)

$$n_{--}=4\pi(2mkTh^{-2})^{\frac{3}{2}}I_{\frac{1}{2}}(\eta_-). \qquad (26)$$

Introducing a function $\phi_-(x)$ defined by

$$\eta_-=\theta^{-1}(4\epsilon)^{-2}(\phi_-/x), \qquad (27)$$

[5] See, for example, Feynman, Metropolis, and Teller, Phys. Rev. 75, 1561 (1949), Sec. V.

[6] These are the usual Thomas-Fermi units except that e has been replaced by λe.

[7] J. McDougall and E. C. Stoner, Trans. Roy. Soc. (London), 237A, 67 (1938).